Generalized symmetries & Anomalies

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Introduction

Anomalies are violations of symmetries.

Anomalies are powerful tools:

- information **invariant** under RG flow (UV–IR), **'t Hooft matching conditions**;
- they obstruct gauging, **constrain the spectrum** of gauge theories
  $\rightarrow$ ABJ for the SM, String Theory, … .

Recently:

- **continuous** and **discrete** symmetries treated in the same way;
- **higher-form symmetries**, the charged objects are strings, branes, etc.;
- **non-invertible symmetries**.

Anomalies and symmetries $\rightarrow$ **description–agnostic**: *e.g.* quarks and gluons vs mesons

Note

In a UV-complete theory with gravity, **NO global symmetries**:

$\rightarrow$ *e.g.* charge falling into a black hole $\rightarrow$ current not conserved.

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Plan

1. Symmetries in Quantum Mechanics
2. Continuous symmetries in Field Theory
3. Symmetries and Defects
4. Anomalies
5. Non-invertible symmetries
6. Higher Form Symmetries
7. Conclusions

Symmetries in Quantum Mechanics

A **Symmetry** is a transformation that:

- maps states $\in \mathbb{P}\mathcal{H}$ into states $\in \mathbb{P}\mathcal{H}$;
  $\rightarrow$ (**Wigner's theorem**) can be represented on $\mathcal{H}$ as an (anti−)unitary(*) operator $\hat{U}$.
- does **NOT** act **trivially** on observables;
  $\rightarrow$ gauge redundancies: **NOT** symmetries!
- **commutes with the Hamiltonian** $\forall \rho \in \mathbb{P}\mathcal{H}$.
  $\rightarrow \left[\hat{U}, H\right] = 0.$

$\Rightarrow$ Observables: valued in **linear**, **faithful** representations of the group G.

G: symmetry group of $\mathbb{P}\mathcal{H}$, **NOT** of $\mathcal{H}$.

Example: Degenerate N−level system

$$\mathcal{H} = \mathbb{C}^N \qquad \Rightarrow \qquad G = \cancel{U(N)} = \frac{U(N)}{\mathcal{Z}(U(N))} = \frac{SU(N)}{\mathbb{Z}_N}$$

In particular, for a **qubit** $(N = 2) \rightarrow$ symmetry group $\boldsymbol{G = SO(3) = SU(2)/\mathbb{Z}_2}$!

(*) I will not discuss the anti-unitary case.

Continuous symmetries in Field Theory

Classically $\varphi(x) \overset{\varepsilon}{\mapsto} \varphi'(x') = \mathcal{F}[\varphi](x)$: infinitesimal transformation:

$\rightarrow$ **conserved current** $J_\mu^{(\varepsilon)}(x)$ via Noether's procedure;

$\rightarrow$ **conserved charge** $Q = \int \mathrm{d}^d x\, J_0^{(\varepsilon)} = \int_{t=t_0} \star J^{(\varepsilon)}$;

$\rightarrow$ Q **generates** the infinitesimal transformation: $\delta_\varepsilon O = \{O, Q\}$.

Quantum mechanically, $\{\cdot,\cdot\} \mapsto [\cdot,\cdot]$ (infinitesimal form of UOU^{-1}).

$J^{(\varepsilon)}$ conservation is replaced by **Ward identities**.

Under **boosts**, a charge which lives at $t = \mathrm{const.}$ acts at different times.

$\rightarrow$ we need a covariant description.

Natural way: Euclidean signature and use

$$\boxed{Q\left(\mathcal{M}_{D-1}\right) = \int_{\mathcal{M}_{D-1}} \star J^{(\varepsilon)}}$$

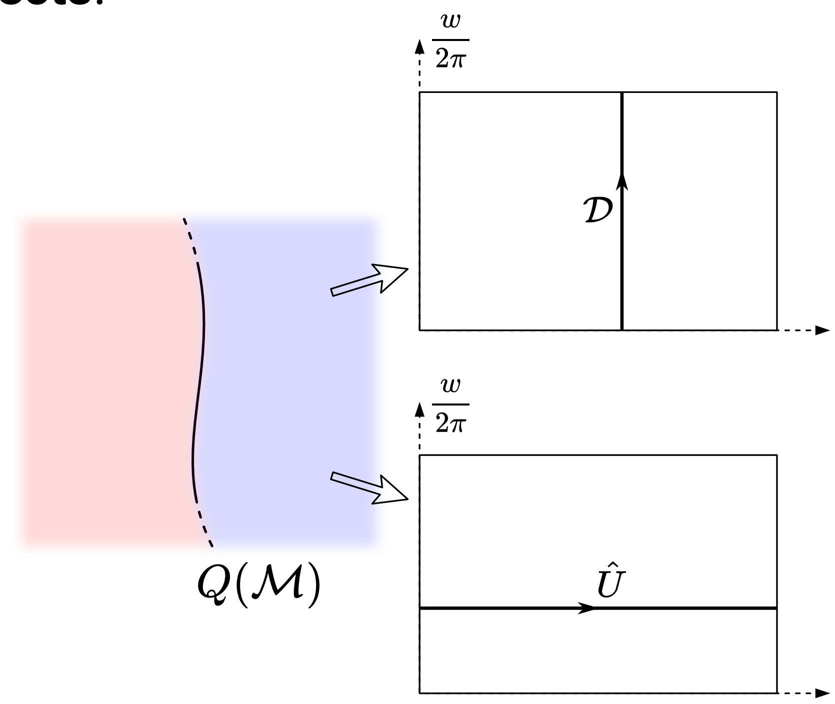
Symmetries and Defects

From Ward identities, charges are **topological objects**.

→ Symmetry ⇒ Topological operator

Generalizing:

Topological operator ⇔ Symmetry



→ A topological operator may **NOT** correspond to a current (discrete symmetries)

Note

A **topological object** can be used as:

- an **operator** on $\mathcal{H}$, if it lives on a space–slice;
- a **defect**, that modifies the quantization, leading to a twisted Hilbert space $\mathcal{H}_{\text{twist}}$.

Anomalies

System $\mathcal{Z}$ with G symmetry group. Turn on a background gauge field A.

→ **'t Hooft anomaly** if

$$\mathcal{Z}[A] \overset{g\in G}{\longmapsto} \exp(i\omega[A,g]) \cdot \mathcal{Z}[A^g]$$

System $\mathcal{Z}$ with symmetry $G_1 \times G_2$. Turn on a background gauge field A_1.

→ **mixed anomaly** if

$$\mathcal{Z}[A_1] \overset{g\in G_2}{\longmapsto} \exp(i\omega[A_1,g]) \cdot \mathcal{Z}[A_1^g]$$

(in both cases, if ω **NOT removable via local counter-terms** in $\Delta S[A]$ in A)

Note

→ anomaly is **invariant under RG–flow** (see Adler–Bardeen theorem, Index theory).

→ if $\mathcal{T}_{\text{UV}}$ has an anomaly, then $\mathcal{T}_{\text{IR}}$ **must** have the same anomaly.
(The converse may not hold)

Example 0: Trivial Anomaly

$$\mathcal{H} = \mathbb{C}^2 \qquad \hat{H} = \sigma^3 \qquad t \sim t + 2\pi \in S^1$$

$G = SO(2) \to$ represented by $U(\alpha) = e^{i\alpha\frac{\sigma^3}{2}}$. Current: $J_0 = \frac{\sigma^3}{2}$.

Turn on **background gauge field**: $A_0(t) = \frac{\alpha}{2\pi}$

$$\mathcal{Z}[A] = \mathrm{tr}\left[e^{-2\pi i\hat{H}}\exp\left(i\int_0^{2\pi} A_0 J_0\,\mathrm{d}t\right)\right] = 2\cos\left(\frac{1}{2}\int_0^{2\pi}\mathrm{d}t\,A_0(t)\right)$$

Gauge transformation: $A^g = A + \partial\Lambda^g$ Let $\Lambda^g(t) = t$:

$$\mathcal{Z}[A^g] = -\mathcal{Z}[A]$$

Is this an anomaly? **No**: we can redefine the charge (= add a **local counter-term**):

$$U'(\alpha) = \exp\left(i\alpha\frac{\sigma^3 + \mathbb{1}}{2}\right) \quad \to \quad \mathcal{Z}'[A^g] = \mathcal{Z}'[A] = 2\cos\left(\frac{1}{2}\int_0^{2\pi}\mathrm{d}t\,A_0(t)\right)e^{\frac{i}{2}\int_0^{2\pi}\mathrm{d}t\,A_0(t)}$$

Example 1: the Qubit

$$\mathcal{H} = \mathbb{C}^2 \qquad \hat{H} = 0 \qquad t \sim t + 2\pi \in S^1$$

$G = SO(3) \to$ represented by $\quad U(\alpha) = e^{i\alpha^a \frac{\sigma^a}{2}}$. $\quad$ Currents: $J_0^{(a)} = \frac{\sigma^a}{2}$.

Turn on **background gauge field** $A_0 = A_0^{(3)} \frac{\sigma^3}{2}$:

$$\mathcal{Z}[A] = \mathrm{tr}\left[e^{-2\pi i \hat{H}} U\left(\int_0^{2\pi} A_0(t)\, \mathrm{d}t \right) \right] = 2\cos\left(\frac{\alpha}{2}\right)$$

Gauge transformation: $\quad A^g = \Omega^g (A + i\partial) \Omega^{g^{-1}} \quad$ Let $\Omega^g = e^{it\frac{\sigma^3}{2}}$:

$$\mathcal{Z}[A^g] = -\mathcal{Z}[A]$$

Maybe the anomaly is removable via a counter−term: $\frac{\sigma^3}{2} \mapsto \frac{\sigma^3 + \mathbb{1}}{2}$.

Let now $\Omega^h = e^{i\pi\frac{\sigma^1}{2}}$: $\quad \mathcal{Z}'[A^h] = \mathcal{Z}'[A] e^{-i\alpha}$

We can move the phase, but we cannot remove it $\to SO(3)$ has an anomaly!

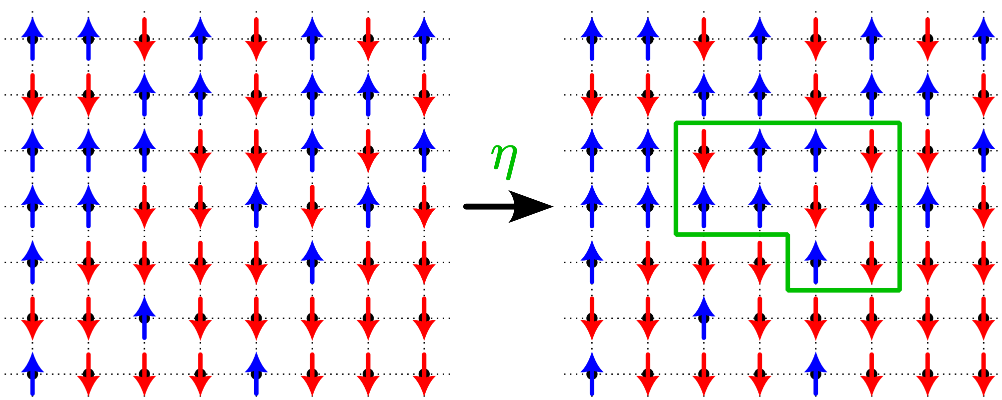
(The anomaly lives in the subgroup $< e^{i\pi\frac{\sigma^1}{2}}, e^{i\pi\frac{\sigma^2}{2}} >= \mathbb{Z}_2 \times \mathbb{Z}_2 \subset SO(3)$)

Example 2: the Critical Ising model — (1)

$$\mathcal{Z} = \sum_{\{\,\text{conf.}\}} e^{-\beta H[\text{conf.}]} \qquad H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j$$

There is a $G = \mathbb{Z}_2$ global symmetry: $\sigma_i \mapsto -\sigma_i$

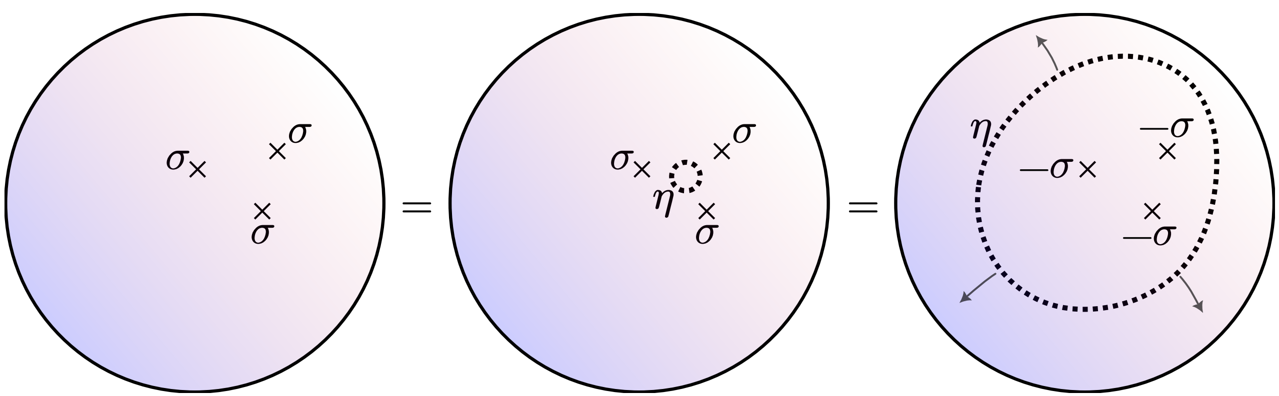
$\rightarrow$ $(D-1)$−dimensional **topological** defect $\eta[\gamma]$



$$\mathcal{Z}[\eta] = \sum_{\{\,\text{conf.}\}} \eta\, e^{-\beta H[\text{conf.}]}$$

$$= \mathcal{Z}[\mathbb{1}]$$

$\rightarrow$ **selection rules**



$$\langle \sigma\sigma\sigma \rangle = -\langle \sigma\sigma\sigma \rangle = 0$$

Example 2: the Critical Ising model — (2)

$$\boxed{D = 2 \qquad m = 3, \quad c = \frac{1}{2}}$$

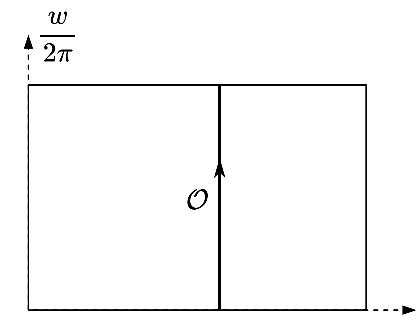
Three **primaries**:

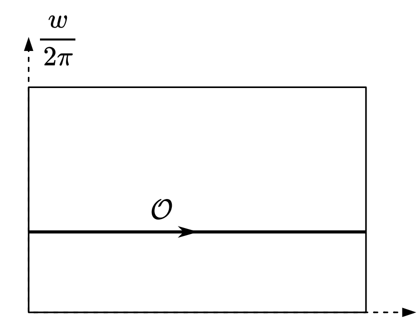
$\mathbb{1}(0,0) \qquad \varepsilon\left(\tfrac{1}{2},\tfrac{1}{2}\right) \qquad \sigma\left(\tfrac{1}{16},\tfrac{1}{16}\right)$

$\left[\mu\left(\tfrac{1}{16},\tfrac{1}{16}\right)\right.$ in twisted sector$\left.\right]$

Fusion rules:

$$[\sigma][\sigma] = [\mathbb{1}] + [\varepsilon]$$
$$[\sigma][\varepsilon] = [\sigma]$$
$$[\varepsilon][\varepsilon] = [\mathbb{1}]$$





$$\mathrm{Tr}\left[\mathcal{O}\, q^{L_0 - \frac{c}{24}}\, \bar{q}^{\overline{L}_0 - \frac{c}{24}}\right] = \lambda_0\, |\chi_0(\tau)|^2 + \lambda_{\frac{1}{2}}\, |\chi_{\frac{1}{2}}(\tau)|^2 + \lambda_{\frac{1}{16}}\, |\chi_{\frac{1}{16}}(\tau)|^2$$
$$= \sum_{i,j} n_{ij} \chi_i\left(-\tfrac{1}{\tau}\right) \overline{\chi}_j\left(-\tfrac{1}{\bar{\tau}}\right)$$

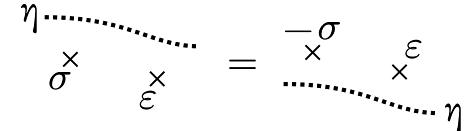
$\rightarrow$

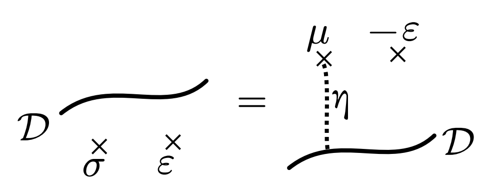
	$\boldsymbol{\lambda_0}$	$\boldsymbol{\lambda_{\frac{1}{2}}}$	$\boldsymbol{\lambda_{\frac{1}{16}}}$
1	1	1	1
η ($\mathbb{Z}_2$)	1	1	-1
$\mathcal{D}$ (KW)	$\sqrt{2}$	$-\sqrt{2}$	0

$\rightarrow$

$$\mathcal{O}|i\rangle = \lambda^i_{\mathcal{O}}\, |i\rangle$$
$$\mathcal{D}\cdot\mathcal{D} = 1 + \eta$$
$$\mathcal{D}\cdot\eta = \mathcal{D}$$
$$\eta\cdot\eta = 1$$

$\rightarrow$



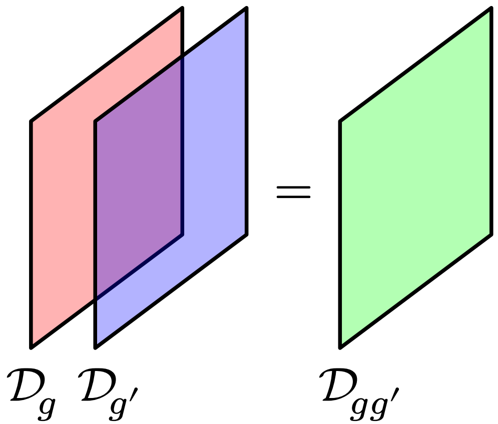


New selection rule: $\mathcal{D} \rightarrow \langle\varepsilon\varepsilon\varepsilon\rangle = -\langle\varepsilon\varepsilon\varepsilon\rangle = 0$

Non-invertible symmetries

Usual invertible symmetry G:

$$\mathcal{D}_g(\Sigma) \times \mathcal{D}_{g'}(\Sigma) = \mathcal{D}_{gg'}(\Sigma), \qquad \mathcal{D}_g(\Sigma) \times \mathcal{D}_{g^{-1}}(\Sigma) = \mathbb{1}$$



We can also define sum of defects:

$$\Big\langle \Big(\mathcal{D}_1 + \mathcal{D}_2\Big)(\cdot) \Big\rangle = \Big\langle \mathcal{D}_1(\cdot) \Big\rangle + \Big\langle \mathcal{D}_2(\cdot) \Big\rangle \quad \rightarrow \quad \mathcal{H}_{1+2} = \mathcal{H}_1 \oplus \mathcal{H}_2$$

Note

We can take $k\mathcal{D}_1 + q\mathcal{D}_2$ with $k, q \in \mathbb{N}$, but **NOT** $\mathcal{D}_1 - \mathcal{D}_2, \quad \frac{1}{7}\mathcal{D}_1$!

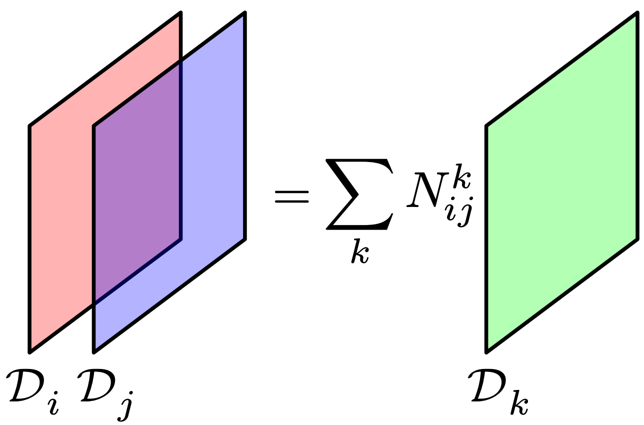
Ordinary global symmetry $\Rightarrow$ Topological defect.

What about $\Leftarrow$? It does not hold in general:

$$\mathcal{D}_i \times \mathcal{D}_j = \sum_k N_{ij}^k \mathcal{D}_k, \qquad N_{ij}^k \in \mathbb{N}$$

[*e.g.* Critical Ising: $\mathcal{D} \times \mathcal{D} = 1 + \eta$]

$\rightarrow$ Fusion rules are **NOT group−like**!



Example 3: XY model — (1)

$$\hat{H} = \sum_i \left[\frac{U}{2}\Pi_i^2 + \frac{J}{2}\left(\Phi_{i+1} - \Phi_i - 2\pi\, n_{i,i+1}\right)^2 - g\cos\left(E_{i,i+1}\right) \right]$$
$$\left[\Phi_i, \Pi_j\right] = i\delta_{i,j} \qquad \left[n_{i,i+1}, E_{j,j+1}\right] = i\delta_{i,j}$$

$G_{\mathrm{UV}} = U(1)_{\mathrm{m}} \rtimes \mathbb{Z}_2$: $\quad Q_{\mathrm{m}} = \sum_i \Pi_i \qquad \mathbb{Z}_2 : \Phi \mapsto -\Phi, n \mapsto -n.$

But with fine−tuning: $g = 0 \quad \to \quad Q_{\mathrm{w}} = -\sum_i n_{i,i+1} \quad \to \quad G_{\mathrm{UV}} = U(1)_{\mathrm{m}} \times \boldsymbol{U(1)_{\mathrm{w}}} \rtimes \mathbb{Z}_2$

$\to$ There is a **mixed anomaly** between $U(1)_{\mathrm{m}}$ and $U(1)_{\mathrm{w}}$.

Ansatz of IR theory: **Compact Boson**: $G_{\mathrm{IR}} = U(1)_{\mathrm{m}} \times U(1)_{\mathrm{w}} \rtimes \mathbb{Z}_2^{\mathrm{R}}$ with mixed anomaly.

Matching coefficients: $\quad \dfrac{J}{U} = \dfrac{R^4}{4\pi^2 \ell_s^4} + \ldots \qquad E_i = \theta(x)$

$$\mathcal{L} = \frac{R^2}{4\pi\ell_s^2}\left(\partial_\mu\varphi\right)^2 = \frac{1}{2\pi}\partial_t\varphi\;\partial_x\theta - \frac{1}{4\pi}\left[\frac{\ell_s^2}{R^2}\left(\partial_x\theta\right)^2 + \frac{R^2}{\ell_s^2}\left(\partial_x\varphi\right)^2\right] = \frac{\ell_s^2}{4\pi R^2}\left(\partial_\mu\theta\right)^2$$
$$\left[\varphi \sim \varphi + 2\pi, \theta \sim \theta + 2\pi\right]$$

Example 3: XY model — (2)

$$\boxed{\mathcal{L} = \frac{\ell_s^2}{4\pi R^2}\left(\partial_\mu\theta\right)^2 + g\cos\theta \qquad [\varphi\sim\varphi+2\pi,\theta\sim\theta+2\pi]}$$

The theory flows to the Compact Boson?

$\rightarrow$ If $U(1)_{\mathrm{w}}$ imposed with fine tuning ($g=0$)

$\rightarrow$ If $U(1)_{\mathrm{w}}$−charged objects are irrelevant

If $g=0$, **conformal symmetry** $\rightarrow$
$$\begin{cases} L_0 = \frac{1}{4}\left(\frac{\ell_s}{R}Q_{\mathrm{m}} + \frac{R}{\ell_s}Q_{\mathrm{w}}\right)^2 + \text{osc}\ldots \\ \overline{L}_0 = \frac{1}{4}\left(\frac{\ell_s}{R}Q_{\mathrm{m}} - \frac{R}{\ell_s}Q_{\mathrm{w}}\right)^2 + \text{osc}\ldots \end{cases}$$

$$\boxed{\begin{aligned}\Delta &= L_0 + \overline{L}_0 = \\ &= \frac{1}{2}\left(\frac{\ell_s}{R}Q_{\mathrm{m}}\right)^2 + \frac{1}{2}\left(\frac{R}{\ell_s}Q_{\mathrm{w}}\right)^2 + \text{osc}\ldots\end{aligned}}$$

$\cos\theta$ relevant $\Leftrightarrow$ $\hat{\Delta}\cos\theta < 2$ $\Leftrightarrow$ $R/\ell_s < 2$ $\Leftrightarrow$ $J/U \lesssim 0.4$ **BKT transition**

($\cos\theta$ irrelevant $\Rightarrow U(1)_{\mathrm{w}}$ anomaly $\Rightarrow$ no trivial ground state)

Example 4: ABJ Anomaly — MASSLESS QED

$$\boxed{\mathcal{L} = i\overline{\Psi}\partial\!\!\!/\Psi \qquad \rightarrow \qquad \mathcal{L} = i\overline{\Psi}D\!\!\!\!/\,\Psi - \frac{1}{4e^2}F^{\mu\nu}F_{\mu\nu}}$$

Symmetries:

Vectorial $U(1)_\mathrm{V}$

$$\begin{cases} \Psi \mapsto \left(e^{+i\Lambda_\mathrm{V}}\right)\Psi \\ \overline{\Psi} \mapsto \overline{\Psi}\left(e^{-i\Lambda_\mathrm{V}}\right) \end{cases}$$

$$J_\mu^{(\mathrm{V})} = -\overline{\Psi}\gamma_\mu\Psi$$

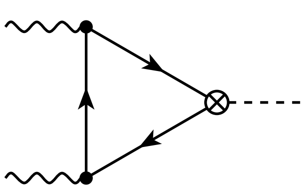
Axial $U(1)_\mathrm{A}$

$$\begin{cases} \Psi \mapsto \left(e^{+i\Lambda_\mathrm{A}\gamma_5}\right)\Psi \\ \overline{\Psi} \mapsto \overline{\Psi}\left(e^{+i\Lambda_\mathrm{A}\gamma_5}\right) \end{cases}$$

$$J_\mu^{(\mathrm{A})} = -\overline{\Psi}\gamma_\mu\gamma_5\Psi$$

After the $U(1)_\mathrm{V}$−gauging:

$$\left\langle \partial_\mu J^{(A)\mu} \right\rangle =$$



$$= -\frac{1}{\mathcal{Z}}\int \delta\left(\mathcal{D}\Psi\mathcal{D}\overline{\Psi}\right)e^{i\mathcal{S}} = \frac{[\hbar]}{16\pi^2}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$$

$$= \partial_\mu\left[\frac{1}{8\pi^2}\varepsilon^{\mu\nu\rho\sigma}A_\nu F_{\rho\sigma}\right]$$ local but **NOT** gauge-invariant!

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Example 5: SM & Non–Abelian Anomalies

Higgs field: $H = \begin{pmatrix} \Phi^+ \\ \Phi^0 \end{pmatrix}$, with $\langle H \rangle = \begin{pmatrix} 0 \\ v \end{pmatrix}, \; v \in \mathbb{R}^+$

Chiral Fermions: (× 3 families)

- leptons $\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, e^c_L$
- quark $\begin{pmatrix} u \\ d \end{pmatrix}^\alpha_L, u^{c,\alpha}_L, d^{c,\alpha}_L \qquad \alpha = 1, 2, 3$

Symmetries: $G_{\text{SM}} = U(1)_{\text{Y}} \times SU(2)_{\text{w}} \times SU(3)_{\text{s}}$

We want to gauge $G_{\text{SM}} \to G$ must **NOT** have anomalies.

Non–Abelian anomalies → **Gauge invariance**: no triangle ⇒ no anomaly (∀ family) (Wess–Zumino consistency conditions determine the rest)

$$(Y^3): \quad \left[2 \cdot \underbrace{\left(-\tfrac{1}{2}\right)}_{(\nu,e)_L}{}^3 + \underbrace{(1)^3}_{e^c_L} + 6 \cdot \underbrace{\left(\tfrac{1}{6}\right)}_{(u,d)^\alpha_L}{}^3 + 3 \cdot \underbrace{\left(-\tfrac{2}{3}\right)}_{u^c_L}{}^3 + 3 \cdot \underbrace{\left(\tfrac{1}{3}\right)}_{d^c_L}{}^3\right] = 0$$

$$([SU(2)]^2 Y): \quad 2 \cdot \left[\underbrace{\left(-\tfrac{1}{2}\right)}_{(\nu,e)_L} + 3 \cdot \underbrace{\left(\tfrac{1}{6}\right)}_{(u,d)_L}\right] = 0 \qquad ([SU(3)]^2 Y): \quad 3 \cdot \left[2 \cdot \underbrace{\left(\tfrac{1}{6}\right)}_{(u,d)_L} + \underbrace{\left(-\tfrac{2}{3}\right)}_{u^c_L} + \underbrace{\left(\tfrac{1}{3}\right)}_{d^c_L}\right] = 0$$

SM must be consistent with perturbative GR → also mixed grav. anomaly must be zero:

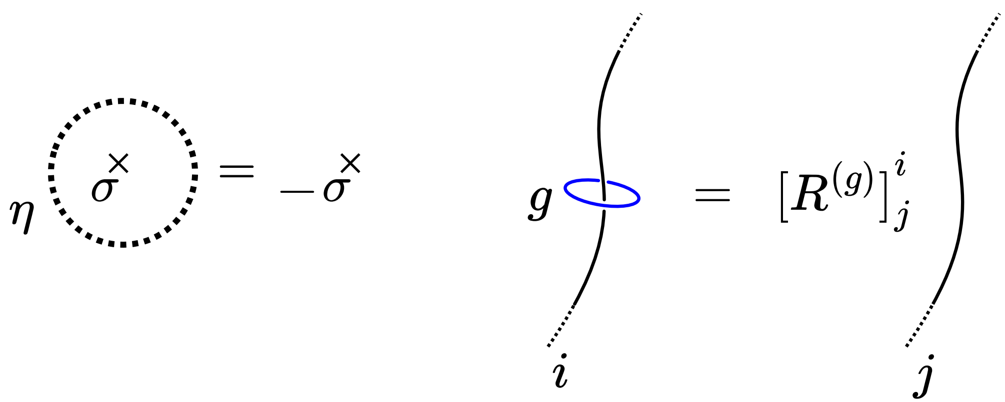
$$(T^2 Y): \quad \left[2 \cdot \underbrace{\left(-\tfrac{1}{2}\right)}_{(\nu,e)_L} + \underbrace{(1)}_{e^c_L} + 6 \cdot \underbrace{\left(\tfrac{1}{6}\right)}_{(u,d)_L} + 3 \cdot \underbrace{\left(-\tfrac{2}{3}\right)}_{u^c_L} + 3 \cdot \underbrace{\left(\tfrac{1}{3}\right)}_{d^c_L}\right] = 0$$

Higher Form Symmetries

Ordinary (0–form) symmetry: $(D-1)$–topological op. $\hookrightarrow$ pointlike op.

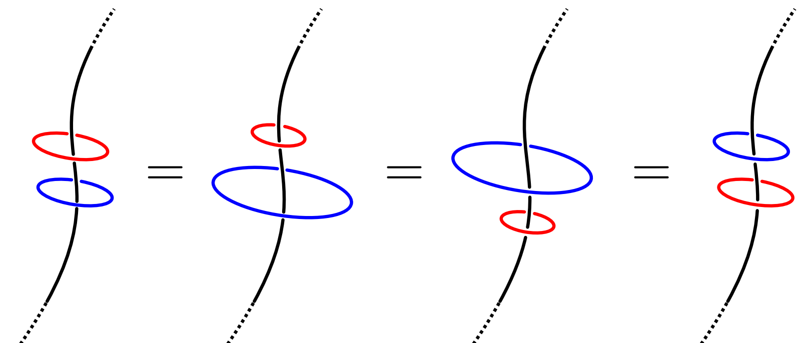
p–form symmetry: $(D-p-1)$–topological op. $\hookrightarrow$ p–op.

$D=3$



They are **Abelian**

$D=3$



Example: **Yang–Mills phase diagram from Wilson Loop**

Wilson loop $\mathcal{W}_n[\gamma] = \mathrm{tr}\left[\mathcal{P}\exp\left(in\int_\gamma A\cdot \mathrm{d}\gamma\right)\right]$

$\to$ **Area Law**: $\left\langle \mathcal{W}_n \right\rangle \sim e^{-A} \sim 0 \to$ unbroken phase $\to$ **confinement** $(V \gtrsim r)$

$\to$ **Perimeter Law**: $\left\langle \mathcal{W}_n \right\rangle \sim e^{-P} \nsim 0 \to$ broken phase $\to V \sim \mathrm{const.} \to$ **Higgs phase**

$\to$ **Coulomb Law**: $\left\langle \mathcal{W}_n \right\rangle > e^{-P} \nsim 0 \to$ broken phase $\to V \lesssim \frac{1}{r} \to$ **Coulomb phase**

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Example 6: Maxwell theory in $D = 4$

Without matter: two $U(1)^{(1)}$ symmetries:

Electric 2−form current

$$J_\mathrm{e} = \frac{2}{e^2}\star F$$

$\hookrightarrow$ Wilson line $\mathcal{W} = \mathcal{P}e^{in\int_\gamma A}$

Magnetic 2−form current

$$J_\mathrm{m} = \frac{1}{2\pi}F$$

$\hookrightarrow$ 't Hooft line $\mathcal{T} = \mathcal{P}e^{in\int_\gamma \tilde{A}}$

With n−charged matter field: $U(1)_\mathrm{e} \mapsto \mathbb{Z}_n$

$\to$ They have a **mixed 't Hooft anomaly**:

$$\frac{1}{2\pi}\int \star B_\mathrm{m}\wedge F \quad\to\quad \star J = \frac{1}{2\pi}\,\mathrm{d}\star B_\mathrm{m} \quad\to\quad \mathrm{d}J_\mathrm{e} = \frac{1}{2\pi}\,\mathrm{d}\star B_\mathrm{m}$$

Both symmetries are **spontaneously broken**: $\langle\mathcal{W}\rangle$ and $\langle\mathcal{T}\rangle$ follows **Coulomb law**.

$\to$ The Goldstone boson is the photon:

$$\langle 0|J_\mathrm{e}^{\mu\nu}(x)|\epsilon,p\rangle = (\epsilon^\mu p^\nu - \epsilon^\nu p^\mu)e^{ipx}$$

Example 7: Non–Abelian Gauge theories

Higher−form symmetries tell us about the **global structure of the group**.

As in the abelian case

→ **Electric 1–form symmetry** $\hookrightarrow$ Wilson lines

→ **Magnetic 1–form symmetry** $\hookrightarrow$ 't Hooft lines

But we cannot use all Wilson and 't Hooft lines: **GNO quantization** ($\sim$ Dirac quant.)

$$\boxed{\vec{m} \cdot \vec{\mu} = 2\pi\mathbb{Z}}$$

Two significant cases with $\mathfrak{su}(N)$ algebra:

$$G = SU(N) \quad \rightarrow \quad G_{\mathrm{e}}^{(1)} = \mathbb{Z}_N \quad \text{and} \quad G_{\mathrm{m}}^{(1)} = \mathbb{1}$$

$$G = \mathbb{P}SU(N) = SU(N)/\mathbb{Z}_N \quad \rightarrow \quad G_{\mathrm{e}}^{(1)} = \mathbb{1} \quad \text{and} \quad G_{\mathrm{m}}^{(1)} = \mathbb{Z}_N$$

Conclusions

Anomalies and Generalized Symmetries are **features** of the theory.

Anomalies:

- RG invariant
- obstruct gaugings
- constrain spectra

Generalized Symmetries:

- phase transitions via new order parameters
- can have anomalies
- can be gauged

They establish connections among various disciplines:

- Mathematics (**Index theory, algebraic topology** and **category theory**)
- **String Theory** and **Conformal Field Theories**
- **Condensed Matter Physics**

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Thank you for your attention!

Defects, Twisted b.c., Flat Connections

Example: complex boson on the circle ($x \sim x + 2\pi$)

$$\mathcal{L} = \frac{g}{4\pi}\partial_\mu \Phi^\dagger \partial^\mu \Phi \qquad \mathcal{H} = \frac{\pi}{4g}\,|\Pi|^2 + \frac{g}{4\pi}\,|\partial_x \Phi|^2$$

$$\to\ \Phi(x,t) = \sum_{n\neq 0} \frac{1}{\sqrt{2gE_n}}\left\{a_n e^{in(x-t)} + b_n^\dagger e^{-in(x+t)}\right\} + \dot{\Phi}_0 t \qquad E_n = |n|, n \in \mathbb{Z}$$

→ **twisted boundary conditions**

$$\Phi(x+2\pi) = e^{i\theta}\Phi(x) \quad \to \quad E_n = |n| \qquad n^{(a)} \in \mathbb{Z} + \frac{\theta}{2\pi} \qquad n^{(b)} \in \mathbb{Z} - \frac{\theta}{2\pi}$$

→ **flat background gauge field**

$$A_\mu = \left(0, \frac{\theta}{2\pi}\right) \quad \to \quad n \in \mathbb{Z} \qquad E_n^{(a)} = |n - A_x| \qquad E_n^{(b)} = |n + A_x|$$

→ **topological defect**

$$J_\mu = i(\Pi_\mu \Phi - \Pi_\mu^\dagger \Phi^\dagger), \quad \left\langle (\cdot) e^{i\theta \int \mathrm{d}t\, J_x} \right\rangle = \int \mathcal{D}\Phi \mathcal{D}\Phi^\dagger (\cdot) e^{i\int \mathcal{L}_{\text{eff}}} \qquad \mathcal{L}_{\text{eff}} = \mathcal{L}_0 + \theta\ \delta(x) J_x$$

Twisted boundary conditions $\Leftrightarrow$ **Flat background gauge field** $\Leftrightarrow$ **Topological defect**

Higher Form Symmetries — Comparison

Ordinary (0–form) symmetries | **Extension to p–form symmetries**

Currents (1–forms): → Higher-spin currents ($(p+1)$–forms):

$$J = J_\mu \, \mathrm{d}x^\mu \qquad J = \frac{1}{p!} J_{\mu_1 \dots \mu_{p-1}} \, \mathrm{d}x^{\mu_1} \wedge \dots \wedge \mathrm{d}x^{\mu_{p-1}}$$

Charges:

$$Q(\Sigma_{D-p}) = \int_{\Sigma_{D-p}} \mathrm{d}\star J^{(p+1)} = \int_{\partial\Sigma_{D-p}} \star J^{(p+1)}$$

Ward Identities:

$$\langle Q(\partial\Sigma_{D-p}) O[\mathrm{T}_p] \rangle = -i \, \mathrm{Link}(\partial\Sigma_{D-p}, \mathrm{T}_p) \langle \delta O[\mathrm{T}] \rangle$$

$$\langle U_g(\partial\Sigma_{D-p}) O[\mathrm{T}_p] \rangle = \mathcal{R}\left(g^{\mathrm{Link}(\partial\Sigma_{D-p}, \mathrm{T}_p)}\right) \langle O[\mathrm{T}_p] \rangle$$

Continuous formulation:

→ Transformation parameter: ξ^p with $\mathrm{d}\xi = 0$ $\qquad \delta\mathcal{S} = \int \frac{1}{p!} J_{\mu_1 \dots \mu_p} \, \mathrm{d}\xi^{\mu_1 \dots \mu_p}$

→ Equation of the motion: $\qquad \mathrm{d}\star J = 0$

→ Poincaré dual: $\Sigma_{D-p-1} \mapsto \xi^p$ closed form → $\qquad \left\langle Q(\xi) \mathcal{O}_p\left(\mathcal{M}\right) \right\rangle = -i \int_{\mathcal{M}} \xi \cdot \langle \delta O \rangle$